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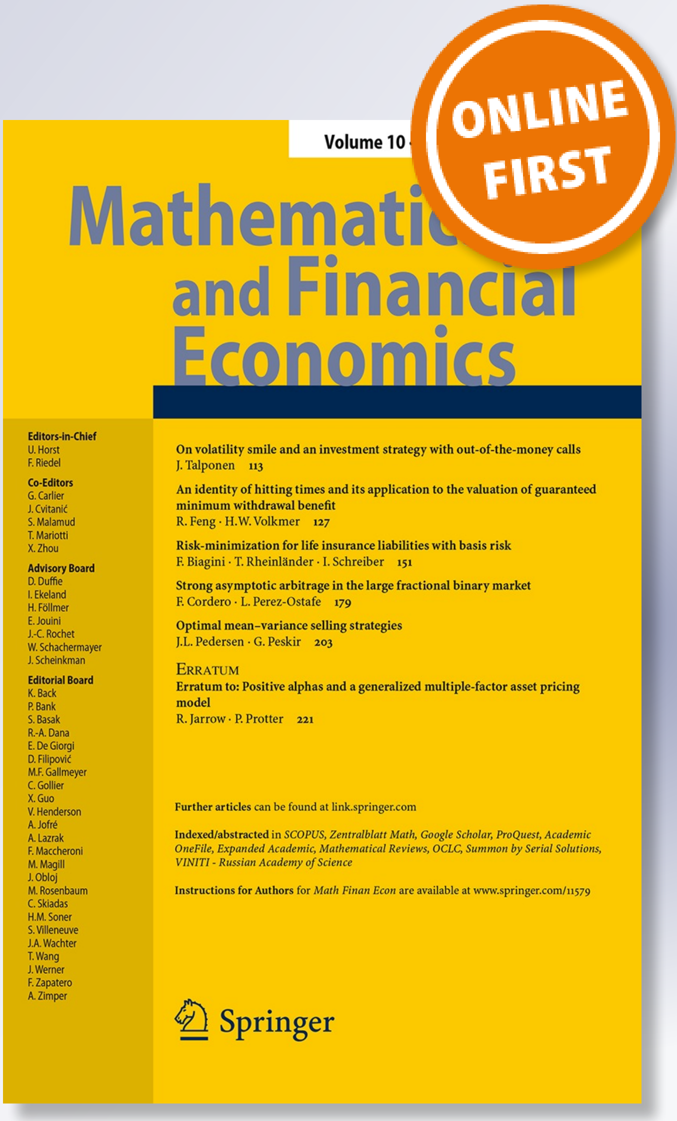

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On the equivalence of financial structures with long-term assets

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Abstract In a stochastic financial exchange economy, two financial structures are equivalent if, for each given state price, the marketable payoffs are identical for the associated asset prices. The key property of two equivalent financial structures is that, when associated with any standard exchange economy, they lead to the same financial equilibrium. We exhibit a sufficient condition for the equivalence of two financial structures without re-trading with possibly long-term assets. We then apply this result to financial structures built upon primitive assets and their re-trading. We also borrow an assumption from Bonnisseau and Chéry (Ann Financ 10:523–552, 2014) to prove the equivalence between a financial structure and its reduced forms.

Keywords Equivalent financial structures · Financial equilibrium · Multi-period model · Long-term assets · Financial sub-structure · Reduced forms

JEL Classification D5 · D4 · G1

1 Introduction

We consider stochastic financial exchange economies defined on a given finite date-event tree representing time and uncertainty. The financial structures may include long-term assets.

We study the equivalence relation on financial structures introduced in [4,7], when the portfolios of agents are unconstrained. Two financial structures are equivalent if, for each

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state price, the marketable payoffs are the same for the arbitrage free asset prices associated to the given state price. In other words, this means that the ranges of the full payoff matrices are identical.

A financial structure allows economic agents to transfer wealth across nodes of the date-event tree through the marketable payoff set. Thereby given a spot price p, the budget set of an agent is fully determined by the marketable payoff set. So the budget sets are the same for two equivalent financial structures. The main consequence is that, regardless of the standard exchange economy Σ, the existence of a financial equilibrium in Σ associated with a financial structure $\mathcal{F}$ is equivalent to the existence of equilibrium in Σ associated with any other financial structure $\mathcal{F}'$ belonging to the equivalence class of $\mathcal{F}$. Furthermore, the equilibrium consumption and the equilibrium spot price are the same, only the asset portfolios have to be suitably modified. Hence the importance of studying the notion of equivalence between the financial structures since the existence of a financial equilibrium for a given financial structure is extended to the equivalence class of this financial structure.

Equivalent financial structures have been studied, among others, by Aouani-Cornet and Cornet-Ranjan [3,7] in the two-period case. In [7], it is proved that two financial structures are equivalent when the ranges of their payoff matrices are equal. We have generalized this result to the multi-period case if all assets are short-term in [5]. By means of examples in Sect. 4, we show that, with long-term assets, equality between the images of payoff matrices of two financial structures is neither necessary nor sufficient to get the equivalence of these structures.

To describe a financial structure, we follow the exposition in Angeloni-Cornet [1] where an asset is issued at a given node, called issuance node and never re-traded afterward. Indeed, this approach actually encompasses the case often considered in the literature where the financial structure is built upon primitive assets issued at some issuance nodes providing payoffs for the future periods and then re-traded at the successive periods. See Magill-Quinzii [8] for a complete description. The re-trading of an asset can actually be interpreted as the issuance of a new asset with the payoff being the truncation of the payoffs of the initial asset for the successors of the re-trading node. But the Angeloni-Cornet's approach also encompasses the case where some assets are re-traded at some nodes but not at all nodes after the initial issuance node. As proved in [1] and using the terminology of Sect. 3, a financial structure with re-trading is equivalent to a financial structure without re-trading. So, since the equivalence of financial structures is a transitive relation, we provide first a general result, Proposition 4.1, for the equivalence of financial structures without re-trading and then we apply it to the case of financial structures with re-trading.

So, to get the equivalence with long-term assets for a structure without re-trading, we introduce an additional assumptions, Assumptions **R1**, on the payoff matrices of two financial structures. Precisely, Assumption **R1** means that, at each emission node ξ, the assets issued at node ξ for the two structures offer the same possibilities of transfer for the successors of ξ. In other words, the marketable payoffs generated by the assets issued at the same node are identical for the two structures. For a two-period economy, this just means that the ranges of the payoff matrices are the same since there is a unique emission node. The main result of the article is that Assumption **R1** is sufficient to get the equivalence of the financial structures. Nevertheless, note that Assumption **R1** is not necessary to get the equivalence as illustrated in an example in Sect. 4.

We apply this result to the case of a financial structures with re-trading where assets are re-traded at every node after their issuance node like in Magill-Quinzii [8]. In this case, note that a financial structure is fully described by the payoff matrix of the primitive assets. To do the link with the previous model, following [1], we introduce the re-trading extension of the financial structure by considering that the re-trade of an asset is equivalent to issuing a new asset. We prove that if the primitive financial structures satisfy Assumption **R1**, then the re-trading extensions also satisfy Assumption **R1**, so the financial structures with re-trading are equivalent.

Then, we study the equivalence of a financial structure with their reduced forms. A reduced form is obtained by removing the redundant assets. This concept is extensively studied in the two period case in [2,4]. The interest for studying this question comes from the methodology to prove the existence of a financial equilibrium. Indeed, we need a fixed point argument requiring the compactness of attainable portfolios. With a financial structure, we may not have bounded attainable portfolios due to the presence of redundant assets. So, a way to get an equilibrium is: considering a reduced form by removing redundant assets; obtaining bounded attainable portfolios for the reduced form; proving the existence of an equilibrium for the reduced form; getting an equilibrium for the original economy by equivalence.

We provide an example showing that a structure may not be equivalent with their reduced form. To get the equivalence, we borrow Assumption **R** from [6]. A financial structure satisfies Asssumption **R** if the returns of the assets issued at a node ξ are not redundant with the returns of the assets issued previously. We show that a financial structure satisfying this assumption is equivalent to their reduced forms.

In Sect. 2, we describe the general framework of a financial exchange economy and we define a financial equilibrium. In Sect. 3, we state the definition of equivalence between two financial structures and we state the result on the link between financial equilibrium for two equivalent financial structures. In Sect. 4, we present and comment our key assumption **R1**, we prove the equivalence under **R1** and then we develop the applications for the re-trading case and the reduced forms.

2 Financial exchange economy and equilibrium

In this section, we present the model and the notations, which are borrowed from Angeloni-Cornet [1] and are essentially the same as those of Magill-Quinzii [8].

2.1 Time and uncertainty

We[1] consider a multi-period exchange economy with $(T+1)$ dates, $t \in \mathcal{T} := \{0, ..., T\}$, and a finite set of agents $\mathcal{I}$. The uncertainty is described by a date-event tree $\mathbb{D}$ of length $T+1$. The set $\mathbb{D}_t$ is the set of nodes (also called date-events) that could occur at date t and the family $(\mathbb{D}_t)_{t\in\mathcal{T}}$ defines a partition of the set $\mathbb{D}$; for each $\xi \in \mathbb{D}$, we denote by $t(\xi)$ the unique date $t \in \mathcal{T}$ such that $\xi \in \mathbb{D}_t$.

At date $t = 0$, there is a unique node ξ_0, that is $\mathbb{D}_0 = \{\xi_0\}$. As $\mathbb{D}$ is a tree, each node ξ in $\mathbb{D} \setminus \{\xi_0\}$ has a unique immediate predecessor denoted $pr(\xi)$ or ξ^-. The mapping pr

1 We use the following notations. A $(\mathbb{D} \times \mathcal{J})$-matrix A is an element of $\mathbb{R}^{\mathbb{D}\times\mathcal{J}}$, with entries $(a^j_\xi)_{(\xi\in\mathbb{D}, j\in\mathcal{J})}$; we denote by $A_\xi \in \mathbb{R}^{\mathcal{J}}$ the ξ-th row of A and by $A^j \in \mathbb{R}^{\mathbb{D}}$ the j-th column of A. We recall that the transpose of A is the unique $(\mathcal{J} \times \mathbb{D})$-matrix ${}^t\text{A}$ satisfying $(Ax) \bullet_{\mathbb{D}} y = x \bullet_{\mathcal{J}} ({}^t\text{Ay})$ for every $x \in \mathbb{R}^{\mathcal{J}}$, $y \in \mathbb{R}^{\mathbb{D}}$, where $\bullet_{\mathbb{D}}$ [resp. $\bullet_{\mathcal{J}}$] denotes the usual inner product in $\mathbb{R}^{\mathbb{D}}$ [resp. $\mathbb{R}^{\mathcal{J}}$]. We denote by rank$A$ the rank of the matrix A and by Vect (A) the range of A, that is the linear sub-space spanned by the column vectors of A. For every subset $\tilde{\mathbb{D}} \subset \mathbb{D}$ and $\tilde{\mathcal{J}} \subset \mathcal{J}$, the matrix $A^{\tilde{\mathcal{J}}}_{\tilde{\mathbb{D}}}$ is the $(\tilde{\mathbb{D}} \times \tilde{\mathcal{J}})$-sub-matrix of A with entries a^j_ξ for every $(\xi, j) \in (\tilde{\mathbb{D}} \times \tilde{\mathcal{J}})$. Let x, y be in $\mathbb{R}^n$; $x \geq y$ (resp. $x \gg y$) means $x_h \geq y_h$ (resp. $x_h > y_h$) for every $h = 1, \ldots, n$ and we let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$, $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x \gg 0\}$. We also use the notation $x > y$ if $x \geq y$ and $x \neq y$. The Euclidean norm in the different Euclidean spaces is denoted $\|.\|$ and the closed ball centered at x and of radius $r > 0$ is denoted $\bar{B}(x, r) := \{y \in \mathbb{R}^n \mid \|y - x\| \leq r\}$.

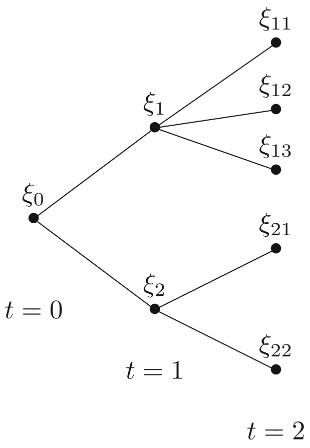

Fig. 1 The tree $\mathbb{D}$

maps $\mathbb{D}_t$ to $\mathbb{D}_{t-1}$. Each node $\xi \in \mathbb{D} \setminus \mathbb{D}_T$ has a set of immediate successors defined by $\xi^+ = \left\{\bar{\xi} \in \mathbb{D} : \xi = \bar{\xi}^-\right\}$.

For $\tau \in \mathcal{T} \setminus \{0\}$ and $\xi \in \mathbb{D} \setminus \cup_{t=0}^{\tau-1}\mathbb{D}_t$, we define $pr^\tau(\xi)$ by the recursive formula: $pr^\tau(\xi) = pr\left(pr^{\tau-1}(\xi)\right)$. We then define the set of successors and the set of predecessors of ξ as follows:

$$\mathbb{D}^+(\xi) = \left\{\xi' \in \mathbb{D} : \exists \tau \in \mathcal{T} \setminus \{0\} \mid \xi = pr^\tau\left(\xi'\right)\right\}$$
$$\mathbb{D}^-(\xi) = \left\{\xi' \in \mathbb{D} : \exists \tau \in \mathcal{T} \setminus \{0\} \mid \xi' = pr^\tau(\xi)\right\}$$

For each $\xi \in \mathbb{D}$, we note by $\mathbb{D}(\xi)$ the union of ξ with $\mathbb{D}^+(\xi)$. If $\xi' \in \mathbb{D}^+(\xi)$ [resp. $\xi' \in \mathbb{D}(\xi)$], we use the notation $\xi' > \xi$ [resp. $\xi' \geq \xi$]. Note that $\xi' \in \mathbb{D}^+(\xi)$ if and only if $\xi \in \mathbb{D}^-\left(\xi'\right)$ and similarly $\xi' \in \xi^+$ if and only if $\xi = \left(\xi'\right)^-$.

A simple example

Let $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}, \xi_{13}, \xi_{21}, \xi_{22}\}$, as in Fig. 1, $T = 2$, the length of $\mathbb{D}$ is 3, $\mathbb{D}_2 = \{\xi_{11}, \xi_{12}, \xi_{13}, \xi_{21}, \xi_{22}\}$, $\xi_1^+ = \{\xi_{11}, \xi_{12}, \xi_{13}\}$, $\mathbb{D}^+(\xi_2) = \{\xi_{21}, \xi_{22}\}$, $t(\xi_{11}) = t(\xi_{12}) = t(\xi_{13}) = t(\xi_{21}) = t(\xi_{22}) = 2$, $\mathbb{D}^-(\xi_{11}) = \{\xi_0, \xi_1\}$.

2.2 The financial structure

At each node $\xi \in \mathbb{D}$, there is a spot market on which a finite set $\mathbb{H} = \{1, \ldots, H\}$ of divisible and physical goods are exchanged. We assume that each good is perishable, that is, its life does not last more than one date. In this model, a commodity is a pair (h, ξ) of a physical good $h \in \mathbb{H}$ and the node $\xi \in \mathbb{D}$ at which the good is available. Then the commodity space is $\mathbb{R}^{\mathbb{L}}$, where $\mathbb{L} = \mathbb{H} \times \mathbb{D}$. An element $x \in \mathbb{R}^{\mathbb{L}}$ is called a consumption, that is to say $x = (x(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{L}}$, where $x(\xi) = (x(h, \xi))_{h \in \mathbb{H}} \in \mathbb{R}^{\mathbb{H}}$ for each $\xi \in \mathbb{D}$.

We denote by $p = (p(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{L}}$ the vector of spot prices and $p(\xi) = (p(h, \xi))_{h \in \mathbb{H}} \in \mathbb{R}^{\mathbb{H}}$ is called the spot price at node ξ. The spot price $p(h, \xi)$ is the price at the node ξ for immediate delivery of one unit of the physical good h. Thus the value of a consumption $x(\xi)$ at node $\xi \in \mathbb{D}$ (measured in unit account of the node ξ) is

$$p(\xi) \bullet_{\mathbb{H}} x(\xi) = \sum_{h \in \mathbb{H}} p(h, \xi)\, x(h, \xi).$$

We describe the financial structure according to the exposition in Angeloni-Cornet [1] where an asset is issued at a given node, called issuance node and never re-traded afterward. Indeed, this approach actually encompasses the case often considered in the literature where the financial structure is built upon primitive assets issued at the issuance nodes providing payoffs for the future periods and then re-traded at the successive periods. See Magill-Quinzii [8] for a complete description. The re-trading of an asset can be interpreted as the issuance of a new asset with the payoff being the truncation of the payoffs of the initial asset for the successors of the re-trading node. As proved in [1] and using the terminology of Sect. 3, a financial structure with re-trading is equivalent to a financial structure without re-trading. So, since the equivalence of financial structures is a transitive relation, we provide first a general result, Proposition 4.1, for the equivalence of financial structures without re-trading and then we apply it to the case of financial structures with re-trading.

The financial structure is constituted by a finite set of assets denoted $\mathcal{J} = \{1, \ldots, J\}$. An asset $j \in \mathcal{J}$ is a contract issued at a given and unique node in $\mathbb{D}$ denoted $\xi(j)$ and called issuance node of j. Each asset is bought or sold only at its issuance node $\xi(j)$ and yields payoffs only at the successor nodes ξ' of $\mathbb{D}^+(\xi(j))$. To simplify the notation, we consider the payoff of asset j at every node $\xi \in \mathbb{D}$ and we assume that it is zero if ξ is not a successor of the issuance node $\xi(j)$. The payoff may depend upon the spot price vector $p \in \mathbb{R}^{\mathbb{L}}$ and is denoted by $V_\xi^j(p)$. Formally, we assume that $V_\xi^j(p) = 0$ if $\xi \notin \mathbb{D}^+(\xi(j))$. An asset is a short term asset if it has a non-zero payoff only at the immediate successors of the issuance node, that is, $V_{\xi'}^j(p) = 0$ if $\xi' \notin \xi^+$. In the following, we consider only *non trivial assets*, that is each asset has a non zero return in at least one node.

$z = (z^j)_{j \in \mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$ is called the portfolio of agent i. If $z^j > 0$ [resp. $z^j < 0$], then $|z^j|$ is the quantity of asset j bought [resp. sold] by agent i at the issuance node $\xi(j)$.

To summarize a financial structure $\mathcal{F} = \left(\mathcal{J}, (\xi(j))_{j \in \mathcal{J}}, V\right)$ consists of

- a set of non trivial assets $\mathcal{J}$,
- a node of issuance $\xi(j)$ for each asset $j \in \mathcal{J}$,
- a payoff mapping $V : \mathbb{R}^{\mathbb{L}} \to \mathbb{R}^{\mathbb{D} \times \mathcal{J}}$ which associates to every spot price $p \in \mathbb{R}^{\mathbb{L}}$ the $(\mathbb{D} \times \mathcal{J})$-payoff matrix $V(p) = \left(V_\xi^j(p)\right)_{\xi \in \mathbb{D}, j \in \mathcal{J}}$ and satisfies the condition $V_\xi^j(p) = 0$ if $\xi \notin \mathbb{D}^+(\xi(j))$.

The price of asset j is denoted by q_j; it is paid at its issuance node $\xi(j)$. We let $q = (q_j)_{j \in \mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$ be the asset price vector.

The full payoff matrix $W(p, q)$ is the $(\mathbb{D} \times \mathcal{J})$-matrix with the following entries:

$$W_\xi^j(p, q) := V_\xi^j(p) - \delta_{\xi, \xi(j)} q_j,$$

where $\delta_{\xi, \xi'} = 1$ if $\xi = \xi'$ and $\delta_{\xi, \xi'} = 0$ otherwise.

So, given the prices (p, q), the full flow of returns for a given portfolio $z \in \mathbb{R}^{\mathcal{J}}$ is $W(p, q)z$ and the full return at node ξ is

$$\begin{aligned}
[W(p,q)z](\xi) &:= W_\xi(p, q) \bullet_{\mathcal{J}} z = \sum_{j \in \mathcal{J}} V_\xi^j(p) z^j - \sum_{j \in \mathcal{J}} \delta_{\xi, \xi(j)} q_j z^j \\
&= \sum_{\{j \in \mathcal{J} \mid \xi(j) < \xi\}} V_\xi^j(p) z^j - \sum_{\{j \in \mathcal{J} \mid \xi(j) = \xi\}} q_j z^j,
\end{aligned}$$

We now recall that for a given spot price p, the asset price q is an arbitrage free price if it does not exist a portfolio $z \in \mathbb{R}^{\mathcal{J}}$ such that $W(p, q)z > 0$. q is an arbitrage free price if and only if it exists a so-called state price vector $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ such that ${}^t W(p, q)\lambda = 0$ (see, e.g.

Magill-Quinzii [8]). Taken into account the particular structure of the matrix $W(p, q)$, this is equivalent to

$$\forall j \in \mathcal{J}, \quad \lambda_{\xi(j)} q_j = \sum_{\xi \in \mathbb{D}^+(\xi(j))} \lambda_\xi V_\xi^j(p).$$

Conversely, for a given state price vector $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$, there exists a unique associated arbitrage free price denoted $q(\lambda)$ satisfying ${}^t W(p, q)\lambda = 0$, which is defined by the above formula.

Some additional notations

We now introduce some additional notations. For all $\xi \in \mathbb{D} \setminus \mathbb{D}_T$, $\mathcal{J}(\xi)$ is the set of assets issued at the node ξ, that is $\mathcal{J}(\xi) = \{j \in \mathcal{J} \mid \xi(j) = \xi\}$ and $\mathcal{J}\left(\mathbb{D}^-(\xi)\right)$ is the set of assets issued at a predecessor of ξ, that is $\mathcal{J}\left(\mathbb{D}^-(\xi)\right) = \{j \in \mathcal{J} \mid \xi(j) < \xi\}$. $\mathbb{D}^e$ is the set of nodes at which there is the issuance of at least one asset, that is, $\xi \in \mathbb{D}^e$ if $\mathcal{J}(\xi) \neq \emptyset$. If $\xi \notin \mathbb{D}^e$, $\mathcal{J}(\xi) = \emptyset$ and, by convention, we let $\operatorname{Im} V^{\mathcal{J}(\xi)}(p) = \{0\}$.

In all our numerical examples, we assume that there is a unique good at each node of the tree and the price of one unit of the good is equal to 1. Consequently, we will denote the payoff matrix (resp. the full payoff matrix) by V (resp. $W(q)$).

2.3 The stochastic exchange economy

We consider a finite set of consumers $\mathcal{I} = \{1, \ldots, I\}$. Each agent $i \in \mathcal{I}$ has a consumption set $X_i \subset \mathbb{R}^{\mathbb{L}}$, which consists of all possible consumptions. An allocation is an element $x \in \prod_{i \in \mathcal{I}} X_i$ and we denote by x_i the consumption of agent i, which is the projection of x on X_i.

The tastes of each consumer $i \in \mathcal{I}$ are represented by a *strict preference correspondence* $P_i : \prod_{j \in \mathcal{I}} X_j \longrightarrow X_i$, where $P_i(x)$ defines the set of consumptions that are strictly preferred to x_i for agent i, given the consumption x_j for the other consumers $j \neq i$. P_i represents the consumer tastes, but also his behavior with respect to time and uncertainty, especially his impatience and attitude toward risk. If consumer preferences are represented by utility functions $u_i : X_i \longrightarrow \mathbb{R}$ for each $i \in \mathcal{I}$, the strict preference correspondence is defined by $P_i(x) = \{\bar{x}_i \in X_i | u_i(\bar{x}_i) > u_i(x_i)\}$.

Finally, for each node $\xi \in \mathbb{D}$, every consumer $i \in \mathcal{I}$ has a node endowment $e_i(\xi) \in \mathbb{R}^{\mathbb{H}}$ (contingent on the fact that ξ prevails) and we denote by $e_i = (e_i(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{L}}$ the endowments for the whole set of nodes. The exchange economy Σ can be summarized by

$$\Sigma = \left[\mathbb{D}, \mathbb{H}, \mathcal{I}, (X_i, P_i, e_i)_{i \in \mathcal{I}}\right].$$

2.4 Financial equilibrium

We now consider a financial exchange economy, which is defined as the couple of an exchange economy Σ and a financial structure $\mathcal{F}$. It can thus be summarized by

$$(\Sigma, \mathcal{F}) := \left[\mathbb{D}, \mathbb{H}, \mathcal{I}, (X_i, P_i, e_i)_{i \in \mathcal{I}}, \mathcal{J}, (\xi(j))_{j \in \mathcal{J}}, V\right].$$

Given the price $(p, q) \in \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$, the budget set of consumer $i \in \mathcal{I}$ is $B^i_{\mathcal{F}}(p, q)$ defined by:[2]

[2] For $x = (x(\xi))_{\xi \in \mathbb{D}}$, $p = (p(\xi))_{\xi \in \mathbb{D}}$ in $\mathbb{R}^{\mathbb{L}} = \mathbb{R}^{\mathbb{H} \times \mathbb{D}}$ (with $x(\xi)$, $p(\xi)$ in $\mathbb{R}^{\mathbb{H}}$) we let $p \Box x = (p(\xi) \bullet_{\mathbb{H}} x(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{D}}$.

$$\left\{ (x_i, z_i) \in X_i \times \mathbb{R}^{\mathcal{J}} : \forall \xi \in \mathbb{D},\, p(\xi) \bullet_{\mathbb{H}} [x_i(\xi) - e_i(\xi)] \leq W_\xi(p, q) \bullet_{\mathcal{J}} z_i \right\}$$

or

$$\left\{ (x_i, z_i) \in X_i \times \mathbb{R}^{\mathcal{J}} :\ p\Box(x_i - e_i) \leq W(p, q)\, z_i \right\}.$$

We now introduce the definition of a financial equilibrium:

Definition 2.1 An equilibrium of the financial exchange economy $(\Sigma, \mathcal{F})$ is a list of strategies and prices $(\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in \left(\mathbb{R}^{\mathbb{L}}\right)^{\mathcal{I}} \times \left(\mathbb{R}^{\mathcal{J}}\right)^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}} \setminus \{0\} \times \mathbb{R}^{\mathcal{J}}$ such that

(a) for every $i \in \mathcal{I}$, $(\bar{x}_i, \bar{z}_i)$ maximizes the preferences P_i in the budget set $B^i_{\mathcal{F}}(\bar{p}, \bar{q})$, in the sense that

$$(\bar{x}_i, \bar{z}_i) \in B^i_{\mathcal{F}}(\bar{p}, \bar{q}) \text{ and } \left[P_i(\bar{x}) \times \mathbb{R}^{\mathcal{J}}\right] \bigcap B^i_{\mathcal{F}}(\bar{p}, \bar{q}) = \emptyset;$$

(b) $\sum_{i\in\mathcal{I}} \bar{x}_i = \sum_{i\in\mathcal{I}} e_i$ and $\sum_{i\in\mathcal{I}} \bar{z}_i = 0$.

We recall that the equilibrium asset price is arbitrage free under the following Non-Satiation Assumption:

Assumption NS (i) *(Non-Saturation at Every Node)* For all $\bar{x} \in \prod_{i\in\mathcal{I}} X_i$ such that $\sum_{i\in\mathcal{I}} \bar{x}_i = \sum_{i\in\mathcal{I}} e_i$, for every $i \in \mathcal{I}$, for every $\xi \in \mathbb{D}$, there exists $x_i \in X_i$ such that, for each $\xi' \neq \xi$, $x_i(\xi') = \bar{x}_i(\xi')$ and $x_i \in P_i(\bar{x})$.

(ii) if $x_i \in P_i(\bar{x})$, then $[x_i, \bar{x}_i[\in P_i(\bar{x})$.

Proposition 2.1 (Magill-Quinzii [8], Angeloni-Cornet [1]) *Under Assumption (NS), if $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of the economy $(\Sigma, \mathcal{F})$ then the asset price $\bar{q}$ is arbitrage free i.e., there exists a state price $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ such that ${}^tW(\bar{p}, \bar{q})\lambda = 0$.*

3 Equivalent financial structures

In this section we will define an equivalence relation on financial structures. We will show that the existence of an equilibrium in an exchange economy associated with a given financial structure is equivalent to the existence of equilibrium in this exchange economy associated with any other financial structure equivalent to the first one. So equivalence allows to extend the existence results for financial equilibrium to a whole class of financial structures. Hence the importance of studying the notion of equivalence between the financial structures.

Definition 3.1 Let $\mathcal{F}_1 = (\mathcal{J}_1, (\xi(j))_{j\in\mathcal{J}_1}, V^1)$ and $\mathcal{F}_2 = (\mathcal{J}_2, (\xi(j))_{j\in\mathcal{J}_2}, V^2)$ be two financial structures. We say that $\mathcal{F}_1$ is equivalent to $\mathcal{F}_2$ with respect to a given spot price p (denoted by $\mathcal{F}_1 \simeq_p \mathcal{F}_2$) if for all state price $\lambda = (\lambda_\xi)_{\xi\in\mathbb{D}} \in \mathbb{R}^{\mathbb{D}}_{++}$, $\operatorname{Im} W^1(p, q^1(\lambda)) = \operatorname{Im} W^2(p, q^2(\lambda))$ where $q^1(\lambda)$ and $q^2(\lambda)$ are the unique arbitrage free prices associated with λ.

We say that $\mathcal{F}_1$ is equivalent to the $\mathcal{F}_2$ if for all spot price vector $p \in \mathbb{R}^{\mathbb{L}}$, $\mathcal{F}_1 \simeq_p \mathcal{F}_2$.

The intuition behind this definition is that the financial structures allow agents to transfer wealth across nodes of the date-event tree. Thereby given a spot price p, their budget set is determined by the set of marketable payoffs that is the range of the full payoff matrix. To be equivalent, two financial structures must provide the same set of marketable payoffs whatever is the state price and the associated arbitrage free asset prices.

Proposition 3.1 *For each spot price $p \in \mathbb{R}^{\mathbb{L}}$, the relation $\simeq_p$ defined above is an equivalence relation.*

The proof is left to the reader.

The main consequence of this definition is given below and states that, regardless of the standard exchange economy Σ, consumption equilibria are the same when agents carry out their financial activities through two different equivalent structures $\mathcal{F}_1$ and $\mathcal{F}_2$.

Proposition 3.2 *Let Σ be an exchange economy satisfying Assumption* ***NS****.*

Let $\mathcal{F}_1 = \left(\mathcal{J}_1, (\xi(j))_{j\in\mathcal{J}_1}, V^1\right)$ and $\mathcal{F}_2 = (\mathcal{J}_2, (\xi(j))_{j\in\mathcal{J}_2}, V^2)$ be two equivalent financial structures.

Let $(\bar{x}, \bar{z}, \bar{p}, \bar{q}^1)$ be an equilibrium of $(\Sigma, \mathcal{F}_1)$. Then there exists $\hat{z}$ and $\bar{q}^2$ such that $(\bar{x}, \hat{z}, \bar{p}, \bar{q}^2)$ is an equilibrium of $(\Sigma, \mathcal{F}_2)$.

Proof of Proposition 3.2 Since $(\bar{x}, \bar{z}, \bar{p}, \bar{q}^1)$ is an equilibrium, Proposition 2.1 implies that $\bar{q}^1$ is an arbitrage free price. So, there exists a state price $\lambda = (\lambda_\xi)_{\xi\in\mathbb{D}} \in \mathbb{R}^{\mathbb{D}}_{++}$ such that ${}^tW^1(\bar{p}, \bar{q}^1)\lambda = 0$. Let $\bar{q}^2$ be the unique arbitrage free price for the financial structure $\mathcal{F}_2$ associed with λ. Since $\mathcal{F}_1 \simeq \mathcal{F}_2$, we have $\operatorname{Im} W^1(\bar{p}, \bar{q}^1) = \operatorname{Im} W^2(\bar{p}, \bar{q}^2)$ (See Definition 3.1).

For all $i \neq 1$, let $\hat{z} \in \mathbb{R}^{\mathcal{J}_2\mathcal{I}}$ be such that $W^1(\bar{p}, \bar{q}^1)\bar{z}_i = W^2(\bar{p}, \bar{q}^2)\hat{z}_i$. Such $\hat{z}_i$ exists because $\operatorname{Im} W^1(\bar{p}, \bar{q}^1) = \operatorname{Im} W^2(\bar{p}, \bar{q}^2)$. Let $\hat{z}_1 = -\sum_{i\in\mathcal{I};i\neq 1} \hat{z}_i$. We now show that $(\bar{x}, \hat{z}, \bar{p}, \bar{q}^2)$ is an equilibrium of $(\Sigma, \mathcal{F}_2)$. Indeed, for all $i \in \mathcal{I}$,

$$W^2(\bar{p}, \bar{q}^2)\hat{z}_i = W^1(\bar{p}, \bar{q}^1)\bar{z}_i$$

This is obvious for $i \neq 1$ and if $i = 1$, as $\sum_{i\in\mathcal{I}} \bar{z}_i = 0$,

$$\begin{aligned} W^2(\bar{p}, \bar{q}^2)\hat{z}_1 &= W^2(\bar{p}, \bar{q}^2)\left(-\sum_{i\in\mathcal{I};i\neq 1} \hat{z}_i\right) = -\sum_{i\in\mathcal{I};i\neq 1}\left[W^2(\bar{p}, \bar{q}^2)\hat{z}_i\right] \\ &= -\sum_{i\in\mathcal{I};i\neq 1}\left[W^1(\bar{p}, \bar{q}^1)\bar{z}_i\right] = W^1(\bar{p}, \bar{q}^1)\left(-\sum_{i\in\mathcal{I};i\neq 1} \bar{z}_i\right) = W^1(\bar{p}, \bar{q}^1)\bar{z}_1 \end{aligned}$$

With this remark, we easily prove that $B^i_{\mathcal{F}_1}(\bar{p}, \bar{q}^1) = B^i_{\mathcal{F}_2}(\bar{p}, \bar{q}^2)$ and $(\bar{x}_i, \hat{z}_i) \in B^i_{\mathcal{F}_2}(\bar{p}, \bar{q}^2)$ for all i, which is enough to conclude since the feasibility conditions are satisfied. □

We now provide some examples of equivalent financial structures. The proofs are given in Appendix.

Example 3.1 (Scalar multiplicator) Let $\mathcal{F} = (\mathcal{J}, (\xi(j))_{j\in\mathcal{J}}, V)$ be a financial structure. For each $\alpha \in \mathbb{R} \setminus \{0\}$, the *$\alpha$-product of $\mathcal{F}$*,

$$\mathcal{F}_\alpha = \left(\mathcal{J}, (\xi(j))_{j\in\mathcal{J}}, V^\alpha = \alpha V\right)$$

is equivalent to $\mathcal{F}$.

Example 3.2 (Union of financial structures) Let $\mathcal{F}_1 = (\mathcal{J}_1, (\xi(j))_{j\in\mathcal{J}_1}, V^1)$ and $\mathcal{F}_2 = (\mathcal{J}_2, (\xi(j))_{j\in\mathcal{J}_2}, V^2)$ be two financial structures. The financial structure[3] $\mathcal{F} := \mathcal{F}_1 \cup \mathcal{F}_2 := (\mathcal{J} := \mathcal{J}_1 \sqcup \mathcal{J}_2, (\xi(j))_{j\in\mathcal{J}}, V = [V^1, V^2])$ is called the *Union of* $\mathcal{F}_1$ and $\mathcal{F}_2$. $\mathcal{F}_1 \cup \mathcal{F}_2 \simeq \mathcal{F}_2 \cup \mathcal{F}_1$ and if $\mathcal{F}_1 \simeq_p \mathcal{F}_2$, then $\mathcal{F}_1 \cup \mathcal{F}_2 \simeq_p \mathcal{F}_1$.

[3] $\mathcal{J}_1 \sqcup \mathcal{J}_2$ is the union of assets of $\mathcal{F}_1$ and of $\mathcal{F}_2$ where, the common assets in $\mathcal{J}_1 \cap \mathcal{J}_2$ are counted twice in the new structure, if $\mathcal{J}_1 \cap \mathcal{J}_2 \neq \emptyset$. The matrix $[V^1, V^2]$ is the $(\mathbb{D} \times (\mathcal{J}_1 \sqcup \mathcal{J}_2))$ matrix whose first $\mathcal{J}_1$ columns are those of V^1 and the last $\mathcal{J}_2$ columns are those of V^2.

Combining the two previous examples we obtain :

Example 3.3 Let $\mathcal{F}_1 = (\mathcal{J}_1, (\xi(j))_{j\in\mathcal{J}_1}, V^1)$ and $\mathcal{F}_2 = (\mathcal{J}_2, (\xi(j))_{j\in\mathcal{J}_2}, V^2)$ be two financial structures such that $\mathcal{F}_1 \simeq_p \mathcal{F}_2$ with respect to the spot price vector p. For each pair $(\alpha, \beta) \in \mathbb{R}^* \times \mathbb{R}^*$, the structure $\mathcal{F}_{\alpha,\beta} := \alpha\mathcal{F}_1 \cup \beta\mathcal{F}_2 = (\mathcal{J} = \mathcal{J}_1 \sqcup \mathcal{J}_2, (\xi(j))_{j\in\mathcal{J}}, V^{\alpha,\beta} = [\alpha V^1, \beta V^2])$ is equivalent to $\mathcal{F}_1$ and to $\mathcal{F}_2$ with respect to p.

Example 3.4 (*Stability of the equivalence by reunion.*) Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ be four financial structures. Let a spot price $p \in \mathbb{R}^{\mathbb{L}}$, such that $\mathcal{F}_1 \simeq_p \mathcal{F}_2$ and $\mathcal{F}_3 \simeq_p \mathcal{F}_4$ then

$$\mathcal{F}_1 \cup \mathcal{F}_3 \simeq_p \mathcal{F}_2 \cup \mathcal{F}_4.$$

Definition 3.2 Let $\mathcal{F} = (\mathcal{J}, (\xi(j))_{j\in\mathcal{J}}, V)$ be a financial structure. We call sub-structure of $\mathcal{F}$ any financial structure $\mathcal{F}' = (\mathcal{J}', (\xi(j))_{j\in\mathcal{J}'}, V')$ such that $\mathcal{J}' \subset \mathcal{J}$ and $V' = V^{\mathcal{J}'}$.

The following proposition is a consequence of Example 3.4.

Proposition 3.3 *Given a spot price* $p \in \mathbb{R}^{\mathbb{L}}$, *let* $\mathcal{F}_1 = (\mathcal{J}_1, (\xi(j))_{j\in\mathcal{J}_1}, V^1)$ *and* $\mathcal{F}_2 = (\mathcal{J}_2, (\xi(j))_{j\in\mathcal{J}_2}, V^2)$ *be two financial structures such that there is a sub-structure* $\mathcal{F}_3$ *of* $\mathcal{F}_2$ *which is equivalent to* $\mathcal{F}_1$ *with respect to a spot price* p. *Then we can complete the structure* $\mathcal{F}_1$ *to get a new financial structure* $\mathcal{F}$ *such that* $\mathcal{F} \simeq_p \mathcal{F}_2$.

4 Sufficient conditions for the equivalence

In this section we provide sufficient conditions on the payoff matrices for the equivalence of financial structure with long-term assets. We first study the case of structures without re-trading and then we apply it to the case of structures with re-trading. In the third subsection, we study the equivalence between a given structure and its reduced forms under an assumption borrowed from [6]. We provide a positive result for structures without re-trading, which cannot be extended to structures with re-trading since it is almost incompatible with our necessary condition.

In the two-period case, two financial structures are equivalent if the images of their payoff matrices are equal, (see [7]). In the multi-period case, if all assets are short-term, we have generalized this result in [5].

In the multi-period case, if there are long-term assets, the equivalence between two financial structure does not imply that the images of payoff matrices of two financial structures are equal (see below Remark 4.1) and equality between the images of payoff matrices of two financial structures does not imply that these two financial structures are equivalent (see below Remark 4.2).

Remark 4.1 Let $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_3\}$, $\mathcal{J}_1 = \left\{j_1^1, j_1^2, j_1^3\right\}$ as in Fig. 2, $\mathcal{J}_2 = \left\{j_2^1, j_2^2, j_2^3\right\}$. $\xi(j_1^1) = \xi(j_2^1) = \xi_0$, $\xi(j_1^2) = \xi(j_2^2) = \xi_1$ and $\xi(j_1^3) = \xi(j_2^3) = \xi_2$. Let $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^4_{++}$ be a state price and $(q_1, q_2) \in \mathbb{R}^{\mathcal{J}_1} \times \mathbb{R}^{\mathcal{J}_2}$ be the couple of arbitrage free prices for the two financial structures associated to λ.

The payoff matrices and the full payoff matrices are:

$$V^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix} \text{ and } W^1(q^1) = \begin{pmatrix} -\frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_0} & 0 & 0 \\ 1 & -\frac{\lambda_2+\lambda_3}{\lambda_1} & 0 \\ 1 & 1 & \frac{-\lambda_3}{\lambda_2} \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix}$$

Fig. 2 The tree $\mathbb{D}$

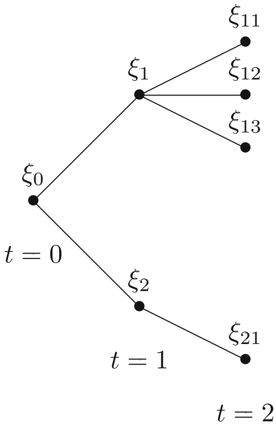

Fig. 3 The tree $\mathbb{D}$

$$V^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix} \text{ and } W^2(q^2) = \begin{pmatrix} -\frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_0} & 0 & 0 \\ 1 & -\frac{\lambda_3}{\lambda_1} & 0 \\ 1 & 0 & -\frac{\lambda_3}{\lambda_2} \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix}$$

It is clear that $\operatorname{rank} W^1(q^1) = \operatorname{rank} W^2(q^2) = 3$, so $\operatorname{Im} W^2(q^2) = \operatorname{Im} W^1(q^1) = \lambda^{\perp}$. So the two structures are equivalent although $\operatorname{Im} V^1 \neq \operatorname{Im} V^2$ because $\operatorname{rank} V^1 = 3 \neq \operatorname{rank} V^2 = 2$.

Remark 4.2 Consider two financial structures such that each contains three assets and $\mathbb{D} = \{\xi_), \xi_1, \xi_2, \xi_{11}, \xi_{12}, \xi_{13}, \xi_{21}\}$ as in Fig. 3. $\xi(j_1^1) = \xi(j_2^1) = \xi(j_2^2) = \xi_0$ and $\xi(j_1^2) = \xi(j_1^3) = \xi(j_2^3) = \xi_1$.

The two payoff matrices V^1 and V^2 are equal, so they have the same image

$$V^1 = V^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{12} \\ \xi_{13} \\ \xi_{21} \end{matrix}$$

With $\lambda = (1, 1, 1, 1, 1, 1, 1)$ and q^1 and q^2 the two associated arbitrage free prices, we have:

$$W^1(q^1)=\begin{pmatrix} \mathbf{-4} & 0 & 0 \\ 1 & \mathbf{-1} & \mathbf{-1} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{12} \\ \xi_{13} \\ \xi_{21} \end{matrix} \text{ and } W^2(q^2)=\begin{pmatrix} \mathbf{-4} & \mathbf{-1} & 0 \\ 1 & 0 & \mathbf{-1} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{12} \\ \xi_{13} \\ \xi_{21} \end{matrix}$$

The two structures are not equivalent since $\mathrm{Im}\, W^1(q^1) \neq \mathrm{Im}\, W^2(q^2)$ even if the two payoff matrices have the same image. Indeed, we can check that the second column vector of the matrix $W^2(q^2)$ does not belong to $\mathrm{Im}\, W^1(q^1)$.

4.1 Equivalence without re-trading under Assumption R1

To get the equivalence, we need an additional assumption on the payoff matrices that we now introduce:

Let $\mathcal{F}_1 = (\mathcal{J}_1, (\xi(j))_{j\in\mathcal{J}_1}, V^1)$ and $\mathcal{F}_2 = (\mathcal{J}_2, (\xi(j))_{j\in\mathcal{J}_2}, V^2)$ be two financial structures defined on the same date-event tree $\mathbb{D}$ and $p \in \mathbb{R}^{\mathbb{L}}$ be a spot price vector.

Assumption R1 $\forall \xi \in \mathbb{D}_1^e \cup \mathbb{D}_2^e$,

$$\mathrm{Im}\, V^{1\,\mathcal{J}_1(\xi)}(p) = \mathrm{Im}\, V^{2\,\mathcal{J}_2(\xi)}(p).$$

Assumption **R1** means that at each emission node ξ, both structures offer the same possibilities of transfer between successor nodes to ξ. In the two-period case, Assumption **R1** at p simply means $\mathrm{Im}\, V^1(p) = \mathrm{Im}\, V^2(p)$ since there is only one emission node, ξ_0, and $\mathrm{Im}\, V^1(p) = \mathrm{Im}\, V^{1\,\mathcal{J}_1(\xi_0)}(p)$. So Assumption **R1** can be seen as the natural extension of the standard assumption on the equality of the range of the payoff matrices when there are more than one issuance node.

Note that since trivial assets are excluded, if Assumption **R1** is satisfied, then the issuance nodes are the same for both financial structures.

Assumption **R1** implies that the images of the two payoff matrices are equal as shown below in the proof of Proposition 4.1. The converse is true when there are only short term assets or when all assets are issued at the same date. Otherwise Assumption **R1** is stronger than assuming the equality of the images of the two payoff matrices. Indeed, in Example 4.3 below, $\mathrm{Im}\, V^1 = \mathrm{Im}\, V^2$ but Assumption **R1** is not satisfied.

We now state the main result of this paper on equivalence with long-term assets when markets are incomplete or not.

Proposition 4.1 *Given a spot price vector $p \in \mathbb{R}^{\mathbb{L}}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two financial structures satisfying Assumption* **R1** *at the spot price $p \in \mathbb{R}^{\mathbb{L}}$. Then $\mathcal{F}_1 \simeq_p \mathcal{F}_2$.*

Proof of Proposition 4.1 Since

$$\mathrm{Im}\, V^1(p) = \sum_{\xi\in\mathbb{D}_1^e} \mathrm{Im}\, V^{1\,\mathcal{J}_1(\xi)}(p) \quad \text{and} \quad \mathrm{Im}\, V^2(p) = \sum_{\xi\in\mathbb{D}_2^e} \mathrm{Im}\, V^{2\,\mathcal{J}_2(\xi)}(p)$$

and since Assumption **R1** implies that $\mathbb{D}_1^e = \mathbb{D}_2^e = \mathbb{D}^e$, one concludes that $\mathrm{Im}\, V^1(p) = \mathrm{Im}\, V^2(p)$ under Assumption **R1**.

Let $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ be a state price and let q^1 and q^2 be the two associated arbitrage free prices. Let $y \in \mathrm{Im}\, W^1(p, q^1)$. There exists $z^1 \in \mathbb{R}^{\mathcal{J}_1}$ such that

$$y = W^1(p, q^1)z^1 = \sum_{\xi \in \mathbb{D}^e} \left[\sum_{j \in \mathcal{J}_1(\xi)} W^{1,j}(p, q^1) z^{1,j} \right].$$

Let $\xi \in \mathbb{D}$ be given. We have:

$$\sum_{j \in \mathcal{J}_1} W^{1,j}_\xi(p, q^1) z^{1,j} = \begin{cases} \sum_{j \in \mathcal{J}_1} V^{1,j}_\xi(p) z^{1,j} \text{ if } \xi \notin \mathbb{D}^e \\ \sum_{j \in \mathcal{J}_1 \setminus \mathcal{J}_1(\xi)} V^{1,j}_\xi(p) z^{1,j} - \sum_{j \in \mathcal{J}_1(\xi)} q^1_j z^{1,j} \text{ if } \xi \in \mathbb{D}^e \end{cases}$$

Since, by Assumption **R1**, for all $\eta \in \mathbb{D}^e$, $\mathrm{Im}\, V^{1\mathcal{J}_1(\eta)}(p) = \mathrm{Im}\, V^{2\mathcal{J}_2(\eta)}(p)$, there exists $z^2 \in \mathbb{R}^{\mathcal{J}_2}$ such that, for all $\eta \in \mathbb{D}^e$,

$$\sum_{j \in \mathcal{J}_1(\eta)} V^{1,j}(p) z^{1,j} = \sum_{j \in \mathcal{J}_2(\eta)} V^{2,j}(p) z^{2,j}.$$

This implies that

$$\sum_{j \in \mathcal{J}_1} W^{1,j}_\xi(p, q^1) z^{1,j} = \begin{cases} \sum_{j \in \mathcal{J}_2} V^{2,j}_\xi(p) z^{2,j} \text{ if } \xi \notin \mathbb{D}^e \\ \sum_{j \in \mathcal{J}_2 \setminus \mathcal{J}_2(\xi)} V^{2,j}_\xi(p) z^{2,j} - \sum_{j \in \mathcal{J}_1(\xi)} q^1_j z^{1,j} \text{ if } \xi \in \mathbb{D}^e \end{cases}$$

But, since ${}^t W^1(p, q^1)\lambda = 0$, with $\xi \in \mathbb{D}^e$,

$$\begin{aligned} \sum_{j \in \mathcal{J}_1(\xi)} q^1_j z^{1,j} &= \sum_{j \in \mathcal{J}_1(\xi)} \left[\sum_{\xi' \in \mathbb{D}^+(\xi)} \frac{\lambda_{\xi'}}{\lambda_\xi} V^{1,j}_{\xi'}(p) \right] z^{1,j} \\ &= \frac{1}{\lambda_\xi} \sum_{\xi' \in \mathbb{D}^+(\xi)} \left[\lambda_{\xi'} \left[\sum_{j \in \mathcal{J}_1(\xi)} V^{1,j}_{\xi'}(p) z^{1,j} \right] \right]. \end{aligned}$$

Since for all $\eta \in \mathbb{D}^e$, $\sum_{j \in \mathcal{J}_1(\eta)} V^{1,j}(p) z^{1,j} = \sum_{j \in \mathcal{J}_2(\eta)} V^{2,j}(p) z^{2,j}$, for each $\xi' \in \mathbb{D}^+(\xi)$

$$\sum_{j \in \mathcal{J}_1(\xi)} V^{1,j}_{\xi'}(p) z^{1,j} = \sum_{j \in \mathcal{J}_2(\xi)} V^{2,j}_{\xi'}(p) z^{2,j}.$$

Consequently, since ${}^t W^2(p, q^2)\lambda = 0$,

$$\begin{aligned} &\frac{1}{\lambda_\xi} \sum_{\xi' \in \mathbb{D}^+(\xi)} \left[\lambda_{\xi'} \left[\sum_{j \in \mathcal{J}_1(\xi)} V^{1,j}_{\xi'}(p) z^{1,j} \right] \right] \\ &= \frac{1}{\lambda_\xi} \sum_{\xi' \in \mathbb{D}^+(\xi)} \left[\lambda_{\xi'} \left[\sum_{j \in \mathcal{J}_2(\xi)} V^{2,j}_{\xi'}(p) z^{2,j} \right] \right] \\ &= \sum_{j \in \mathcal{J}_2(\xi)} \left[\sum_{\xi' \in \mathbb{D}^+(\xi)} \frac{\lambda_{\xi'}}{\lambda_\xi} V^{2,j}_{\xi'}(p) \right] z^{2,j} = \sum_{j \in \mathcal{J}_2(\xi)} q^2_j z^{2,j}. \end{aligned}$$

Hence

$$\sum_{j\in\mathcal{J}_1} W_\xi^{1,j}(p,q^1)z^{1,j} = \begin{cases} \sum_{j\in\mathcal{J}_2} V_\xi^{2,j}(p)z^{2,j} \text{ if } \xi \notin \mathbb{D}^e \\ \sum_{j\in\mathcal{J}_2\setminus\mathcal{J}_2(\xi)} V_\xi^{2,j}(p)z^{2,j} - \sum_{j\in\mathcal{J}_1(\xi)} q_j^2 z^{2,j} \text{ if } \xi \in \mathbb{D}^e \end{cases}$$
$$= \sum_{j\in\mathcal{J}_2} W_\xi^{2,j}(p,q^2)z^{2,j}$$

So for all $\xi \in \mathbb{D}$, $y_\xi = \sum_{j\in\mathcal{J}_1} W_\xi^{1,j}(p,q^1)z^{1,j} = \sum_{j\in\mathcal{J}_2} W_\xi^{2,j}(p,q^2)z^{2,j}$. Consequently $y \in \operatorname{Im}W^2(p,q^2)$ hence $\operatorname{Im}W^1(p,q^1) \subset \operatorname{Im}W^2(p,q^2)$. With a similar reasoning, we can show that $\operatorname{Im}W^2(p,q^2) \subset \operatorname{Im}W^1(p,q^1)$. So $\operatorname{Im}W^2(p,q^2) = \operatorname{Im}W^1(p,q^1)$, that is $\mathcal{F}_1 \simeq_p \mathcal{F}_2$. □

Remark 4.3 Assumption **R1** is not necessary for the equivalence of financial structures. Indeed, we provide now two equivalent financial structures, which do not satisfy Assumption **R1**. Let $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_3\}$ as in Fig. 2, $\mathcal{J}_1 = \{j_1^1, j_1^2, j_1^3\}$, $\mathcal{J}_2 = \{j_2^1, j_2^2, j_2^3\}$. $\xi(j_1^1) = \xi(j_1^2) = \xi(j_2^1) = \xi(j_2^2) = \xi_0$, $\xi(j_1^3) = \xi_2$ and $\xi(j_2^3) = \xi_1$. Let $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ be a state price and let $(q^1, q^2) \in \mathbb{R}^{\mathcal{J}_1} \times \mathbb{R}^{\mathcal{J}_2}$ be the couple of associated arbitrage free prices.

The payoff matrices and the full payoff matrices are:

$$V^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix} \quad \text{and } W^1(q^1) = \begin{pmatrix} -\frac{\lambda_1}{\lambda_0} & \frac{-\lambda_2-\lambda_3}{\lambda_0} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & \frac{-\lambda_3}{\lambda_2} \\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix}$$

$$V^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix} \quad \text{and } W^2(q^2) = \begin{pmatrix} \frac{-\lambda_1-\lambda_3}{\lambda_0} & \frac{-\lambda_2+\lambda_3}{\lambda_0} & 0 \\ 1 & 0 & \frac{\lambda_2}{\lambda_1} \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix}$$

Assumption **R1** is not satisfied since the issuance nodes are not the same for the two financial structures. We have $\operatorname{rank}W^1(q_1) = \operatorname{rank}W^2(q_2) = 3$. Indeed, the rank of the square sub-matrix A_1 (resp. A_2) composed by the three last rows of $W^1(q^1)$ (resp. $W^2(q^2)$) is equal to 3 because $\operatorname{Det}A_1 = 1 + \frac{\lambda_3}{\lambda_2}$ (resp. $\operatorname{Det}A_2 = -1 - \frac{\lambda_2}{\lambda_1}$) which is always different from zero. So, one can conclude that $\operatorname{Im}W^1(q^1) = \operatorname{Im}W^2(q^2) = \lambda^\perp$, hence the financial structures are equivalent.

4.2 Equivalence with re-trading

We deal in this sections with financial structures built upon primitive assets issued at different issuance nodes providing payoffs for the future periods and then re-traded at the successive periods.

Let $\mathcal{F} = \left(\mathcal{J}, (\xi(j))_{j\in\mathcal{J}}, V\right)$ be a financial structure. Suppose that each asset j, once issued, is re-traded at all succeeding nodes except terminal nodes. Each re-traded asset at a node $\xi \in \mathbb{D}^-$ is considered as a *new asset* j_ξ issued at node ξ. The new financial structure thus constituted is called the *re-trading extension* of the primitive financial structure.

Definition 4.1 Let $\mathcal{F} = \left(\mathcal{J}, (\xi(j))_{j\in\mathcal{J}}, V\right)$ be a financial structure. The re-trading of asset $j \in \mathcal{J}$ at node ξ', a successor of $\xi(j)$, denoted $j_{\xi'}$, is the asset issued at ξ', that is, $\xi(j_{\xi'}) = \xi'$, and whose flow of payoffs is given by

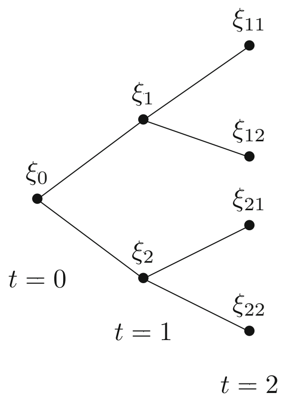

Fig. 4 The tree $\mathbb{D}$

$\tilde{V}_{\xi}^{j_{\xi'}}(p) = V_{\xi}^{j}(p)$, if $\xi \in \mathbb{D}^+(\xi')$;
$\tilde{V}_{\xi}^{j_{\xi'}}(p) = 0$ otherwise.

The re-trading extension of $\mathcal{F}$ is the the new financial structure

$$\left(\tilde{\mathcal{J}}, \left(\xi(j')\right)_{j' \in \tilde{\mathcal{J}}}, \tilde{V}\right),$$

which consists of all primitive assets $j \in \mathcal{J}$ and of all re-trading assets $(j_{\xi'})$ to all nodes $\xi' \in \mathbb{D}^+(\xi(j)) \backslash \{\mathbb{D}_T\}$.

Note that a primitive asset j can be considered as its re-trading at its issuance node that is $j_{\xi(j)} = j$.

Two financial structures with re-trading are equivalent if their re-trading extensions are equivalent. Actually, our result below shows that the information on the primitive assets are enough to conclude about the equivalence.

A simple example

Let $\mathcal{F}$ be a financial structure constituted of two financial assets $\{j^1, j^2\}$ issued at the first date such that

$$V^{j^1} = {}^t(1, 3, -2, 1, 1, 4) \text{ and } V^{j^2} = {}^t(2, -2, 3, -1, 1, 1)$$

and $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}\}$ as in Fig. 4.

The payoff matrix of the re-trading extension $\tilde{\mathcal{F}}$ of $\mathcal{F}$ is:

$$\tilde{V} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 & 0 & 0 \\ -2 & 3 & -2 & 3 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 4 & 1 & 0 & 0 & 4 & 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{12} \\ \xi_{21} \\ \xi_{22} \end{matrix}$$

Notations

For each $\xi' \in \mathbb{D}^-$, we denote by $q_{j_{\xi'}}$ the price of asset $j_{\xi'}$ (i.e., the re-trading of asset j at node ξ'), which is also called the re-trading price of asset j at node ξ'. So, for the financial structure $\tilde{\mathcal{F}}$, both the asset price vector $q = (q_{j_{\xi'}})_{j_{\xi'} \in \tilde{\mathcal{J}}}$ and the portfolio $z = (z^{j_{\xi'}})_{(j_{\xi'}) \in \tilde{\mathcal{J}}}$ now belong to $\mathbb{R}^{\tilde{\mathcal{J}}}$.

Given a spot price $p \in \mathbb{R}^{\mathbb{L}}$ an asset price vector $q \in \mathbb{R}^{\tilde{\mathcal{J}}}$ and a portfolio $z \in \mathbb{R}^{\tilde{\mathcal{J}}}$, the full financial return of z for the financial structure $\tilde{\mathcal{F}}$ at node $\xi \in \mathbb{D}$ is given by:
if $\xi = \xi_0$, $W_{\tilde{\mathcal{F}}\xi}(p, q) \bullet_{\tilde{\mathcal{J}}} z = -\sum_{j \in \mathcal{J} | \xi(j) = \xi_0} q_{j_{\xi_0}} z^{j_{\xi_0}}$
if $\xi \in \mathbb{D}^- \cap \mathbb{D}^+$, $W_{\tilde{\mathcal{F}}\xi}(p, q) \bullet_{\tilde{\mathcal{J}}} z$ is equal to

$$\sum_{j | \xi(j) < \xi} \left(\sum_{\xi(j) \le \xi' < \xi} z^{j_{\xi'}} \right) V_\xi^j(p) - \sum_{j | \xi(j) \le \xi} q_{j_\xi} z^{j_\xi}$$

finally if $\xi \in \mathbb{D}_T$, $W_{\tilde{\mathcal{F}}\xi}(p, q) \bullet_{\tilde{\mathcal{J}}} z$ is equal to:

$$\sum_{j | \xi(j) < \xi} \left(\sum_{\xi(j) \le \xi' < \xi} z^{j_{\xi'}} \right) V_\xi^j(p)$$

So, the full payoff matrix of the previous example is:

$$W_{\tilde{\mathcal{F}}}(\tilde{q}) = \begin{pmatrix} -q_{j^1_{\xi_0}} & -q_{j^2_{\xi_0}} & 0 & 0 & 0 & 0 \\ 1 & 2 & -q_{j^1_{\xi_1}} & -q_{j^2_{\xi_1}} & 0 & 0 \\ 3 & -2 & 0 & 0 & -q_{j^1_{\xi_2}} & -q_{j^2_{\xi_2}} \\ -2 & 3 & -2 & 3 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 4 & 1 & 0 & 0 & 4 & 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{12} \\ \xi_{21} \\ \xi_{22} \end{matrix}$$

We now state a simple condition under which the re-trading extensions of two primitive financial structures satisfy Assumption **R1**.

Proposition 4.2 *Let $p \in \mathbb{R}^{\mathbb{L}}$ be a spot price. Consider two financial structures $\mathcal{F}_1 = \left(\mathcal{J}_1, (\xi(j))_{j \in \mathcal{J}_1}, V^1\right)$ and $\mathcal{F}_2 = \left(\mathcal{J}_2, (\xi(j))_{j \in \mathcal{J}_2}, V^2\right)$. Let $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ be the re-trading extension of $\mathcal{F}_1$ and $\mathcal{F}_2$.*

If $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy Assumption **R1** *at the spot price $p \in \mathbb{R}^{\mathbb{L}}$, then $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ satisfy Assumption* **R1** *at the spot price $p \in \mathbb{R}^{\mathbb{L}}$ and the two financial structures with re-trading $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent at the spot price $p \in \mathbb{R}^{\mathbb{L}}$.*

We remark that the equivalence of the financial structures with re-trading can be checked on the primitive assets since Assumption **R1** is inherited by the re-trading extensions of the financial structures and the equivalence follows from Proposition 4.1.

Proof of Proposition 4.2 Since $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy Assumption **R1** at the spot price $p \in \mathbb{R}^{\mathbb{L}}$, $\mathbb{D}_1^e = \mathbb{D}_2^e = \mathbb{D}^e$. Let ξ be a node. To prove that $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ satisfy Assumption **R1** at the spot price $p \in \mathbb{R}^{\mathbb{L}}$, we have to show that $\operatorname{Im} \tilde{V}^{1\tilde{J}_1(\xi)}(p) = \operatorname{Im} \tilde{V}^{2\tilde{J}_2(\xi)}(p)$. We remark that the assets issued at the date ξ for the structure $\tilde{\mathcal{F}}_k$ $(k = 1, 2)$ is $\tilde{\mathcal{J}}_k(\xi) = \{j_\xi \mid \xi(j) \le \xi\}$. Thus, $\operatorname{Im} \tilde{V}^{k\tilde{J}_k(\xi)}(p) = +_{\xi' \in \mathbb{D}^e | \xi' \le \xi} \operatorname{Im} V^{kJ_k(\xi')}_{\mathbb{D}^+(\xi)}(p)$, where $V^{kJ_k(\xi')}_{\mathbb{D}^+(\xi)}(p)$ is the matrix deduced from

$V^{kJ_k(\xi')}(p)$ by replacing the rows η for $\eta \notin \mathbb{D}^+(\xi)$ by a row with 0 entries. So $\operatorname{Im} V^{kJ_k(\xi')}_{\mathbb{D}^+(\xi)}(p)$ is the orthogonal projection of $\operatorname{Im} V^{kJ_k(\xi')}(p)$ on the subspace $E_{\mathbb{D}^+(\xi)}$ of $\mathbb{R}^{\mathbb{D}}$ defined by $E_{\mathbb{D}^+(\xi)} = \{v \in \mathbb{R}^{\mathbb{D}} \mid \forall \eta \notin \mathbb{D}^+(\xi), v_\eta = 0\}$. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy Assumption **R1** at the spot price p, $\operatorname{Im} V^{1J_1(\xi')}(p) = \operatorname{Im} V^{2J_2(\xi')}(p)$ and, consequently, the orthogonal projections on $E_{\mathbb{D}^+(\xi)}$ are also equal. So, $\operatorname{Im} V^{1J_1(\xi')}_{\mathbb{D}^+(\xi)}(p) = \operatorname{Im} V^{2J_2(\xi')}_{\mathbb{D}^+(\xi)}(p)$. Since this equality holds true for all $\xi' \leq \xi$, one deduces that $\operatorname{Im} \tilde{V}^{1\tilde{J}_1(\xi)}(p) = \operatorname{Im} \tilde{V}^{2\tilde{J}_2(\xi)}(p)$. □

4.3 Equivalence with reduced forms without retrading

In this subsection, we will define the concept of a reduced form of a financial structure. Then we will give a result on the equivalence of a financial structure with its reduced forms. An important motivation for studying the reduced forms of a financial structure is the following.

To get existence of a financial equilibrium, we use a fixed point argument, so we need the compactness of attainable portfolios. With a financial structure, we may have unbounded attainable portfolios due to redundant assets.[4] So, a way to get an equilibrium is:

- considering a reduced form by removing redundant assets;
- then obtaining bounded attainable portfolios for the reduced form;
- then proving the existence for the reduced form;
- by equivalence, getting an equilibrium for the original economy.

Definition 4.2 Let $p \in \mathbb{R}^{\mathbb{L}}$ be a spot price vector. Let $\mathcal{F} = (\mathcal{J}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure. We call a reduced form of $\mathcal{F}$ with respect to p, any financial sub-structure[5] $\mathcal{F}' = (\mathcal{J}', (\xi(j))_{j \in \mathcal{J}'}, V')$ of $\mathcal{F}$ such that $\operatorname{rank} V(p) = \operatorname{rank} V'(p) = \#\mathcal{J}'$.

If for all $p \in \mathbb{R}^{\mathbb{L}}$ we have $\operatorname{rank} V(p) = \operatorname{rank} V'(p) = \#\mathcal{J}'$, we say then that $\mathcal{F}'$ is a *reduced form* of $\mathcal{F}$.

We note that according to Definition 4.2, we obtain a reduced form of a financial structure with respect to a spot price p by simply eliminating the maximum number of redundant assets for p.

Remark 4.4 This example provides a financial structure which is not equivalent to a reduced form. Indeed, let $\mathcal{F}$ be a financial structures with $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_3\}$ as in Fig. 2, $\mathcal{J} = \{j^1, j^2, j^3\}$, $\xi(j^1) = \xi_0$, $\xi(j^2) = \xi_1$ and $\xi(j^3) = \xi_2$. The payoff matrix is

$$V = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix}$$

The matrix of a reduced form $\mathcal{F}'$ of $\mathcal{F}$ is $V' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$.

4 Recall that an asset j_0 of a financial structure $\mathcal{F} = (\mathcal{J}, (\xi(j))_{j \in \mathcal{J}}, V)$ is redundant at the spot price p if the column vector $V^{j_0}(p)$ representing its payoffs on $\mathbb{D}$ is a linear combination of the other column vectors of the matrix $V(p)$.

5 See Definition 3.2.

Let $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^4_{++}$ and $(q, q') \in \mathbb{R}^3 \times \mathbb{R}^2$ the associated arbitrage free prices. We have

$$W(q) = \begin{pmatrix} -\frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_0} & 0 & 0 \\ 1 & -\frac{\lambda_3}{\lambda_1} & 0 \\ 1 & 0 & -\frac{\lambda_3}{\lambda_2} \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix} \text{ and } \operatorname{rank} W(q) = 3$$

while the rank of the full payoff matrix $W'(q')$ is at most equal to two, so $\operatorname{Im} W(q) \neq \operatorname{Im} W'(q')$ hence $\mathcal{F}$ is not equivalent to $\mathcal{F}'$.

The next proposition shows that Assumption **R** borrowed from [6] is a sufficient condition to get the equivalence between a financial structure and its financial sub-structures when the ranges of the payoff matrices are equal.

We first recall the assumption introduced by Bonnisseau and Chéry in [6]. Given a spot price vector $p \in \mathbb{R}^{\mathbb{L}}$, and a payoff matrix $V(p)$.

Assumption R $\forall \xi \in \mathbb{D}^e$,

$$\operatorname{Vect}\left(V_{\mathbb{D}^+(\xi)}^{\mathcal{J}(\mathbb{D}^-(\xi))}(p)\right) \bigcap \operatorname{Vect}\left(V_{\mathbb{D}^+(\xi)}^{\mathcal{J}(\xi)}(p)\right) = \{0\}.$$

Assumption **R** means that the returns of the assets issued at a node ξ are not redundant with the returns of the assets issued at a predecessor node of ξ. So, the issuance of additional assets at ξ are a true financial innovation since the payoffs in the successors of ξ cannot be replicated by the payoffs of a portfolio built with the assets issued before ξ.

Note that when an asset is re-traded, Assumption **R** is typically violated since the new asset exactly replicates the payoffs of the primitive asset, which is issued previously.

Proposition 4.3 *Let $p \in \mathbb{R}^{\mathbb{L}}$ be a spot price vector. Let $\mathcal{F} = (\mathcal{J}, (\xi(j))_{j\in\mathcal{J}}, V)$ be a financial structure satisfying Assumption **R** at the spot price p and let $\mathcal{F}' = (\mathcal{J}', (\xi(j))_{j\in\mathcal{J}'}, V')$ be a financial sub-structure of $\mathcal{F}$. The following assertions are equivalent:*

- **(1)** $\mathcal{F} \simeq_p \mathcal{F}'$;
- **(2)** *For every state price $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$, $\operatorname{Im} W(p, q) = \operatorname{Im} W'(p, q')$ where q and q' are the associated arbitrage free prices;*
- **(3)** *For every state price $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$, $\operatorname{rank} W(p, q) = \operatorname{rank} W'(p, q')$ where q and q' are the associated arbitrage free prices;*
- **(4)** $\operatorname{rank} V(p) = \operatorname{rank} V'(p)$;
- **(5)** $\operatorname{Im} V(p) = \operatorname{Im} V'(p)$;
- **(6)** $\forall \xi \in \mathbb{D}^e$, $\operatorname{Im} V^{\mathcal{J}(\xi)}(p) = \operatorname{Im} V'^{\mathcal{J}'(\xi)}(p)$.

Note that (6) means that $\mathcal{F}$ and $\mathcal{F}'$ satisfy Assumption **R1** at p. So, under Assumption **R**, a sub-structure is equivalent if we remove only redundant assets.

The following corollary is a direct consequence of the previous proposition because we consider a reduced form which is a particular case of a sub-structure and the rank of the payoff matrices are the same by definition.

Corollary 4.1 *Given a spot price vector $p \in \mathbb{R}^{\mathbb{L}}$, let $\mathcal{F} = (\mathcal{J}, (\xi(j))_{j\in\mathcal{J}}, V)$ be a financial structure satisfying Assumption **R** at the spot price p. Then, $\mathcal{F}$ is equivalent with respect to p to each of its reduced forms.*

Since Assumption **R** is always true in the case of a financial structure consisting only of short-term assets, we deduce from Corollary 4.1 the following corollary:

Corollary 4.2 *A financial structure consisting only of short-term assets is equivalent to each of its reduced forms.*

Corollary 4.2 generalizes Proposition 1.5 of the thesis of Aouani [2], which deals with the two-period case. The following corollary is deduced from Corollary 4.1 and Example 3.4.

Corollary 4.3 *Given a spot price vector $p \in \mathbb{R}^{\mathbb{L}}$, let $\mathcal{F}_1 = (\mathcal{J}_1, (\xi(j))_{j\in\mathcal{J}_1}, V^1)$ and $\mathcal{F}_2 = (\mathcal{J}_2, (\xi(j))_{j\in\mathcal{J}_2}, V^2)$ be two financial structure satisfying Assumption **R** at the spot price p. Then for all reduced forms $\mathcal{F}_1'$ of $\mathcal{F}_1$ with respect to p and for all reduced forms $\mathcal{F}_2'$ of $\mathcal{F}_2$ with respect to p we have*

$$\mathcal{F}_1 \cup \mathcal{F}_2 \simeq_p \mathcal{F}_1' \cup \mathcal{F}_2'.$$

Remark 4.5 The following example shows that Assumption **R** is not necessary to get the equivalence between a financial structure and its reduced forms.

Indeed, let $\mathcal{F} = (\mathcal{J}, (\xi(j))_{j\in\mathcal{J}}, V)$ be a financial structures such that $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_3\}$ as in Fig. 2, $\mathcal{J} = \{j^1, j^2, j^3, j^4\}$ and $\xi(j^1) = \xi(j^2) = \xi(j^3) = \xi_0$, $\xi(j^4) = \xi_2$. The payoff matrix is

$$V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix}$$

Assumption **R** is not satisfied since $\text{Vect}\, V_{\mathbb{D}^+(\xi_2)}^{\mathcal{J}(\mathbb{D}^-(\xi_2))} \cap \text{Vect}\, V_{\mathbb{D}^+(\xi_2)}^{\mathcal{J}(\xi_2)} = \mathbb{R} \neq \{0\}$. The financial structure $\mathcal{F}$ has exactly three reduced forms denoted by $\mathcal{F}^1$, $\mathcal{F}^2$ and $\mathcal{F}^3$:

$$V^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, V^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } V^3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The full payoff matrices are as follows:

$$W(.) = \begin{pmatrix} . & . & . & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & . \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$W^1(.) = \begin{pmatrix} . & . & 0 \\ 1 & 0 & 0 \\ 0 & 1 & . \\ 0 & 1 & 1 \end{pmatrix}, W^2(.) = \begin{pmatrix} . & . & . \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } W^3(.) = \begin{pmatrix} . & . & 0 \\ 1 & 0 & 0 \\ 0 & 1 & . \\ 0 & 0 & 1 \end{pmatrix}$$

Let $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^4_{++}$ and $(q, q^1) \in \mathbb{R}^4 \times \mathbb{R}^3$ be the associated arbitrage free prices. Then, $\text{Im}\, W(q) = \text{Im}\, W^1(q^1)$. Indeed, the column vectors of the matrix $W^1(q^1)$ are also column vectors of the matrix $W(q)$ so, $\text{Vect}\, W^1(q^1) \subset \text{Vect}\, W(q)$. Moreover, $\text{rank}\, W^1(q^1) = \text{rank}\, W(q) = 3$. Hence $\mathcal{F}^1$ is equivalent to $\mathcal{F}$. Similarly, we can show that the structures $\mathcal{F}^2$ and $\mathcal{F}^3$ are equivalent to $\mathcal{F}$.

Proof of Proposition 4.3 1) implies 2). By definition of the equivalence of two financial structures. 2) implies 3). Because $\operatorname{rank} W(p,q) = \dim \operatorname{Im} W(p,q)$ and $\operatorname{rank} W'(p,q') = \dim \operatorname{Im} W'(p,q')$. 3) implies 4). Thanks to Assumption **R** (See Proposition 3.5 and Lemma 3.1 in [6]). 4) implies 5). Because by definition of a sub-structure of a financial structure, we have always $\operatorname{Im} V'(p) \subset \operatorname{Im} V(p)$. 5) implies 6). By definition of a sub-structure of a financial structure, we have always for all $\xi \in \mathbb{D}^e$, $\operatorname{Im} V'^{\mathcal{J}'(\xi)}(p) \subset \operatorname{Im} V^{\mathcal{J}(\xi)}(p)$, hence $\operatorname{rank} V'^{\mathcal{J}'(\xi)}(p) \leq \operatorname{rank} V^{\mathcal{J}(\xi)}(p)$. Furthermore, in the proof of Proposition 3.5 in [6], it is shown that Assumption **R** implies $\operatorname{rank} V(p) = \sum_{\xi \in \mathbb{D}^e} \operatorname{rank} V^{\mathcal{J}(\xi)}(p)$. $\mathcal{F}'$ satisfies Assumption **R** as sub-structure of a financial structure satisfying assumption **R**, so, we also have $\operatorname{rank} V'(p) = \sum_{\xi \in \mathbb{D}^e} \operatorname{rank} V'^{\mathcal{J}'(\xi)}(p)$. Hence $\sum_{\xi \in \mathbb{D}^e} \operatorname{rank} V^{\mathcal{J}(\xi)}(p) = \sum_{\xi \in \mathbb{D}^e} \operatorname{rank} V'^{\mathcal{J}'(\xi)}(p)$. Combining with the above inequalities, we conclude that $\operatorname{rank} V'^{\mathcal{J}'(\xi)}(p) = \operatorname{rank} V^{\mathcal{J}(\xi)}(p)$ for all $\xi \in \mathbb{D}^e$, and finally $\operatorname{Im} V'^{\mathcal{J}'(\xi)}(p) = \operatorname{Im} V^{\mathcal{J}(\xi)}(p)$. 6) implies 1). Thanks to Proposition 4.1. □

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Appendix

Proof of Examples of Section 3

Example 3.1 Let p be a spot price, $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ be a state price, α be one non-zero real number and $(q, q^\alpha)\mathbb{R}^{\mathcal{J}} \times \mathbb{R}^{\mathcal{J}}$ be the couple of arbitrage free prices associated to λ. From the arbitrage free condition, $q^\alpha = \alpha q$, so $W^\alpha(p, q^\alpha) = \alpha W(p,q)$, which implies that $\operatorname{Im} W^\alpha(p, q^\alpha) = \operatorname{Im} W(p,q)$ since $\alpha \neq 0$. So, $\mathcal{F} \simeq \mathcal{F}_\alpha$. □

Example 3.2 Let p be a spot price, $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ be a state price and $(q^1, q^2) \in \mathbb{R}^{\mathcal{J}_1} \times \mathbb{R}^{\mathcal{J}_2}$ be the couple of arbitrage free prices associated to λ. From the structure of V, the arbitrage free price q associated to λ for the payoff matrix V is (q^1, q^2). So $W(p,q) = [W^1(p,q^1), W^2(p,q^2)]$. Consequently,

$$\begin{aligned} \operatorname{Im} W(p,q) &= \operatorname{Im} W^1(p,q^1) + \operatorname{Im} W^2(p,q^2) \\ &= \operatorname{Im} W^2(p,q^2) + \operatorname{Im} W^1(p,q^1) \\ &= \operatorname{Im}[W^2(p,q^2), W^1(p,q^1)] \end{aligned}$$

Hence, $\mathcal{F}_1 \cup \mathcal{F}_1 \simeq \mathcal{F}_2 \cup \mathcal{F}_1$. Furthermore, if $\mathcal{F}_1 \simeq \mathcal{F}_2$, then $\operatorname{Im} W^1(p,q^1) = \operatorname{Im} W^2(p,q^2)$ and $\operatorname{Im} W^1(p,q^1) + \operatorname{Im} W^2(p,q^2) = \operatorname{Im} W^1(p,q^1) = \operatorname{Im} W^2(p,q^2)$. So, $\mathcal{F}_1 \cup \mathcal{F}_1 \simeq \mathcal{F}_1 \simeq \mathcal{F}_2$. □

Example 3.4 Let $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ be a state price and $(q^1, q^2, q^3, q^4) \in \mathbb{R}^{\mathcal{J}_1} \times \mathbb{R}^{\mathcal{J}_2} \times \mathbb{R}^{\mathcal{J}_3} \times \mathbb{R}^{\mathcal{J}_4}$ be the arbitrage free prices associated to λ. Like in the previous example, we remark that the full payoff matrix associated to $\mathcal{F}_1 \cup \mathcal{F}_3$ (resp. $\mathcal{F}_2 \cup \mathcal{F}_4$) is $[W^1(p,q^1), W^3(p,q^3)]$ (resp. $[W^2(p,q^1), W^4(p,q^3)]$). Furthermore,

$$\begin{aligned} &\operatorname{Im}[W^1(p,q^1), W^3(p,q^3)] = \operatorname{Im} W^1(p,q^1) + \operatorname{Im} W^3(p,q^3) \text{ and} \\ &\operatorname{Im}[W^2(p,q^2), W^4(p,q^4)] = \operatorname{Im} W^2(p,q^2) + \operatorname{Im} W^4(p,q^4). \end{aligned}$$

Since $\mathcal{F}_1 \simeq_p \mathcal{F}_2$ and $\mathcal{F}_3 \simeq_p \mathcal{F}_4$,

$$\operatorname{Im} W^1(p, q^1) = \operatorname{Im} W^2(p, q^2) \text{ and } \operatorname{Im} W^3(p, q^3) = \operatorname{Im} W^4(p, q^4).$$

Hence $\operatorname{Im}[W^1(p, q^1), W^3(p, q^3)] = \operatorname{Im}[W^2(p, q^2), W^4(p, q^4)]$, which shows that $\mathcal{F}_1 \cup \mathcal{F}_3 \simeq_p \mathcal{F}_2 \cup \mathcal{F}_4$. □

Proof of Proposition 3.3 Let $\mathcal{F}_4$ be the financial structure with the assets of $\mathcal{F}_2$, which are not in $\mathcal{F}_3$, that is, $\mathcal{F}_4 = (\mathcal{J}_4 = \mathcal{J}_2 \setminus \mathcal{J}_3, (\xi(j))_{j \in \mathcal{J}_4}, V^4 = V^{\mathcal{J}_4})$. So $\mathcal{F}_2 \simeq \mathcal{F}_3 \cup \mathcal{F}_4$. Then, for the spot price p, since $\mathcal{F}_1 \simeq_p \mathcal{F}_3$, from Example 3.4, one gets that $\mathcal{F}_1 \cup \mathcal{F}_4 \simeq_p \mathcal{F}_3 \cup \mathcal{F}_4 \simeq_p \mathcal{F}_2$. □

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